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OPTIMIZATION OF MULTIVARIABLE SAMPLED-DATA CONTROL SYSTEMS

by

Jean Lieutaud

Research Report No. PIBMRI-1078-62

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September 1962

MRI

POLYTECHNIC INSTITUTE OF BROOKLYN
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ABSTRACT

This study is concerned with the optimization of multivariable sampled-data control systems. A squared error performance criterion is chosen and the effects of controllability and observability are studied. In particular it is shown that the optimization can always be carried out on a system all of whose coordinates are both controllable and observable. In each case several illustrative examples are included.

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SECTION I

INTRODUCTION

The purpose os this study is to consider a problem of optimization using the point of view developed by Gilbert [1] about the analysis of a multivariable system. In particular the notion of controllability and observability introduced by Kalman [2], modified somewhat by Gilbert [1] will be recalled in order to determine in which way the optimization problem is affected when some of the coordinates are uncontrollable or unobservable.

A sampled - data system will be considered. Section II contains the basic elements of Gilbert's analysis of a multivariable system adapted to the case of a sampled - data. Section III recalls briefly the theory of dynamic programming which is used in the following sections.

Section IV considers a problem of optimization using a quadratic error criterion with given desired final value. Section V treats the regulator problem where the output is desired to follow a given reference input.

It will be shown that the number of initial conditions that has to be stated is equal to the number of observable coordinates. The reference input has to be stated a priori i.e., must be known when the process starts. Finally the optimization can be limited to the part of the system whose coordinates are both controllable and observable.

SECTION II

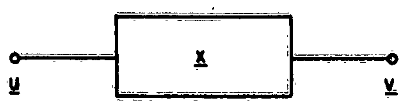
ANALYSIS OF A MULTIVARIABLE SAMPLED-DATA SYSTEM

2.1 Introduction:

A multivariable sampled-data system can be represented by

$$\underline{x}(k+1) = A\,\underline{x}(k) + B\,\underline{u}(k) \qquad (2\text{-}1)$$

$$\underline{y}(k) = C\,\underline{x}(k) + D\,\underline{u}(k) \qquad (2\text{-}2)$$



where $\underline{u}(k)$ is a p dimensional input vector at the sampling instant k

$\underline{v}(k)$ is a q dimensional output vector at the sampling instant k.

FIG. 2-1. GENERAL REPRESENTATION OF A SYSTEM

$\underline{x}(k)$ is a n dimensional state-vector at the instant k. This vector is characteristic of the state of the system at any sampling instant. n is called the order of the system.

A is an nth order square matrix

B is an n rows, p columns matrix

C is a q row, n column matrix

D is a q row, p column matrix

The signal flow-graph of this representation is given in figure 2-2.

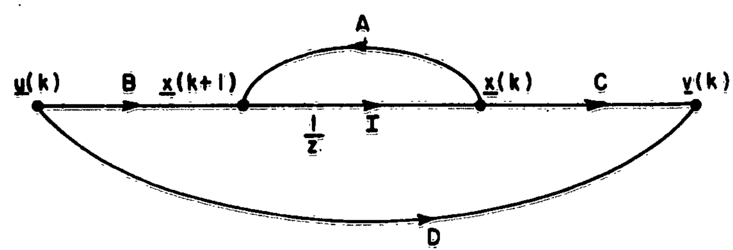


FIG. 2-2. SIGNAL FLOW-GRAPH OF EQUATIONS (2-1) and (2-2)

With this representation the state of the system is known only at the sampling instants. Its behaviour between those instants is not known. In particular, when a problem of optimization will be considered on such a system, determining an optimum input will be interpreted as "determining the values of this input at the sampling instants".

2-2. NORMAL REPRESENTATION OF A SYSTEM [1], [3]

The eigenvalues of matrix A are assumed to be distinct. These eigenvalues are the solutions of the nth order equation

$$\det \left[A - \lambda I\right] = 0 \tag{2-3}$$

where $I = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix}$ is the identity matrix

If λ_i is one particular eigenvalue the vector $\underline{V_i}$ given by the matrix equation

$$\left[A - \lambda_i \;\; I\right] \;\; V_i = 0 \tag{2-4}$$

is called the eigenvector corresponding to the particular eigenvalue λ_i.

If the λ_i are distinct, to each of them corresponds one and only one eigenvector. Thus, there are n different eigenvectors.

Consider now the matrix ρ whose column vectors* are the n eigenvectors, $V_1, V_2, \ldots V_n$. It is possible to show that the matrix

$$\Lambda = \rho^{-1} \; A \rho \tag{2-5}$$

is a diagonal matrix given by

---

*Given a n rows matrix $A = a_{ij}$ the jth column vector $\underline{X_j}$ is the n dimensional vector whose components are the components of the jth column of A.

$$X_j = \begin{bmatrix} a_{1j} \\ a_{ij} \\ \cdot \\ \cdot \\ a_{nj} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \\ 0 & & \lambda_n \end{bmatrix} \tag{2-6}$$

Define now a new set of state-space coordinates by

$$\underline{x} = \rho \, \underline{y} \tag{2-7}$$

The set of equations (2-1) and (2-2) defining initially the system becomes

$$\underline{y}(k+1) = \Lambda \underline{y}(k) + \beta \, u(k) \tag{2-8}$$

$$\underline{v}(k) = \gamma \, y(k) + D \, u(k) \tag{2-9}$$

where

$$\beta = \rho^{-1} B \tag{2-10}$$

$$\gamma = C \, \rho \tag{2-11}$$

The y_i's are called the <u>normal coordinates</u> of the system. The matrix ρ and therefore the normal coordinates are not unique. It is always possible to multiply a column of ρ by a given number or to arrange the columns in another order. By doing so we get another matrix which still " diagonalizes" the matrix A.

The system of equations is stable if Re $\lambda_i < 1$ for all i. The rank of the input $\underline{u}$ is defined as the rank of the matrix B. If this rank is r_u, it means that r_u inputs are independant, the remaining $(p - r_u)$ being linear combinations of the r_u independent inputs.

The system can be considered as having r_u inputs. In the same way, the rank of the output r_v is the rank of the matrix C or γ. It gives the number of linearly independant outputs when $D = 0$.

It is always possible to reduce the number of inputs by $(p - r_u)$ and the number of outputs by $(q - r_v)$ so that the rank of the input (output) is equal to the number of inputs (outputs).

2-3. OBSERVABILITY AND CONTROLLABILITY

These notions were introduced by Kalman [2], and somewhat modified by Gilbert [1].

A system is said to be controllable, if β has no rows of zeros. The controllable coordinates y_i are the ones corresponding to non zero rows of β. The uncontrollable coordinates are the ones corresponding to zero rows of β. It is clear that uncontrollable coordinates cannot be influenced by the input. They only depend on initial conditions or on disturbance inputs.

A system is said to be observable, if γ has no columns of zeros. The observable coordinates are the ones corresponding to non-zero columns of γ. The non-observable coordinates are the ones corresponding to zero columns of γ. It is clear that non-observable coordinates are the ones which do not influence the output.

2-4. CONSEQUENCE

A System S can always be partitioned into four subsystems:

1. A system S^*, all the coordinates of which are both controllable and observable and having a transmission matrix D.
2. A system S^o such that all its coordinates are observable and uncontrollable.
3. A system S^c such that all its coordinates are controllable and unobservable.
4. A system S^f such that all its coordinates are uncontrollable and unobservable.

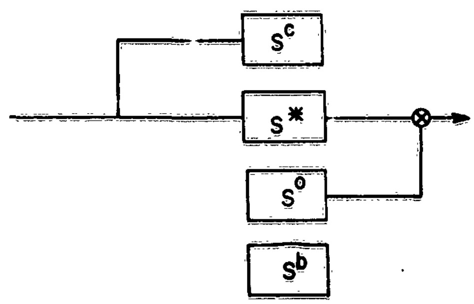


FIG. 2-3. PARTITIONING OF A SYSTEM INTO FOUR SUBSYSTEMS

The systems S^o, S^c, S^f have zero transmission matrices. The proof of this theorem, obtained by partitioning a system given by equations (2-8) and (2-9), is given in appendix I .

2-5 CONCLUSIONS -

A multivariable sampled-data system can be characterized by an n dimensional state-vector. The values of this vector are known only at the sampling instants. The behaviour of the system between those instants is undetermined.

The components of the state-vector are classified into four categories:

1) Some of them are said to be controllable and observable because they are affected by the input of the system and they affect its output.

2) Some components which do not affect the output but are affected by the input are said to be controllable and unobservable.

3) Some components which are affected by the input but do not affect the output are observable and uncontrollable.

4) Some components, which are independant of the input and do not affect the output are uncontrollable and unobservable.

Each group of components of the state vector can be represented as a subsystem of the initial system.

SECTION III

ELEMENTS OF DYNAMIC PROGRAMMING [4] , [5]

3-1 INTRODUCTION -

Consider a multivariable sampled-data system with a p dimensional input vector $\underline{u}$ (k) where k is the kth sampling instant.

FIG. 3-1. GENERAL MULTIVARIABLE SYSTEM

It is desirable to have the output follow as closely as possible a q dimensional reference input vector $\underline{r}$.

The output $\underline{x}$ and the control input $\underline{u}$ are related by the plant dynamics which, in the case of a sampled-data system can be written as

$$\underline{x}\,(k+1) = f\left[\,\underline{x}\,(k),\ \underline{u}\,(k)\right] \tag{3-1}$$

In many cases the possible values of the control inputs are restricted by physical consideration. In our problem it will be assumed that there is no strict limitation on $\underline{u}$ such as $|\underline{u}| \leq A$, where A is a given number. However it will be supposed desirable to have a reasonably small input vector. This will be taken into account by introducing $\underline{u}$ in the performance criterion.

Generally, it is impossible to get the output to have the desired behaviour.

The general performance criterion may be represented in the form

$$S = \int_{t_o}^{T} F(\underline{x},\ \underline{r},\ \underline{u})\,dt \tag{3-2}$$

In the case of a sampled -data system, where the state of the system is known at sampling instants only, this criterion may be replaced by

$$S = \sum_{k=0}^{N} F\left[\underline{x}\,(k),\ \underline{r}\,(k),\ \underline{u}\,(k)\right] \tag{3-3}$$

This is referred to as an N stage process.

3-2 OPTIMAL CONTROL PROBLEM

The problem which is posed now is the following:

Given a performance criterion (2-3) and an arbitrary initial output $\underline{x}(0)$, determine a sequence of control inputs, $\underline{u}^{o}(0), \underline{u}^{o}(1), \ldots \underline{u}^{o}(N-1)$ which minimizes (or maximizes) S.

In the following chapters two types of performance criterion will be considered:

1) Final value problem:

After a certain number of stages the output should be as close as possible to a reference $\underline{r}(k)$. Besides, as we have seen, the limitations on the input is taken into account by introducing the magnitude of $\underline{u}$ into the performance criterion.

The performance criterion will be of the form

$$S = \left[x_1(k) - r_1(k)\right]^2 + \left[x_2(k) - r_2(k)\right]^2 + \ldots + \left[x_q(k) - r_q(k)\right]^2$$

$$+ \sum_{k=0}^{N} u_1^2(k) + u_2^2(k) + \ldots + u_p^2(k) \qquad (3\text{-}4)$$

2) Regulator problem -

Determine $\underline{u}(k)$ which maintains $\underline{x}(k)$ as close as possible to a reference input r(k). In addition $\underline{u}$ must be as "small" as possible.

The criterion for this case is

$$S = \sum_{k=0}^{N} \left[x_1(k) - r_1(k)\right]^2 + \left[x_2(k) - r_2(k)\right]^2 + \ldots \left[x_q(k) - r_q(k)\right]^2$$

$$+ u_1^2(k) + u_2^2(k) + \ldots + u_p^2(k)$$

$$= \sum_{k=0}^{N} \left[\underline{x}(k) - \underline{r}(k)\right]^T \left[\underline{x}(k) - \underline{r}(k)\right] + \underline{u}^T(k)\,\underline{u}(k) \qquad (3\text{-}5)^*$$

---

*The notation A^T is used for the transpose of the matrix A.

It is noticed that expression (3-4) is not exactly of the form of equation (3-3)

Both equations (3-4) and (3-5) can be represented by

$$S = F_o\left[\underline{x}(0), \underline{r}(0), \underline{u}(0)\right] + F_1\left[\underline{x}(1), \underline{r}(1), \underline{u}(1)\right] + \ldots + F_N\left[x(N), r(N), u(N)\right] \tag{3-6}$$

which, for convenience, is expressed as

$$S = F_o(0) + F_1(1) + \ldots + F_N(N) \tag{3-7}$$

In equation (3-5) $F_o \equiv F_1 \equiv \ldots = F_N$

In equation (3-4) $F_o \equiv F_1 = \ldots = F_{N-1} \neq F_N$

3-3 METHOD OF DYNAMIC PROGRAMMING - [5], [2]

1) Principle of optimality - [5]

An optimal sequence of control inputs, $\underline{u}(0)$, $\underline{u}(1)$, ... $\underline{u}(N-1)$, has the property that whatever the initial state $\underline{x}(0)$ and the initial choice of $\underline{u}(0)$ are, the remaining sequence $\underline{u}(1)$, $\underline{u}(2)$, ... $\underline{u}(N-1)$ must constitute an optimal sequence for the N-1 stages process starting at $k = 1$, with the initial state $\underline{x}(1)$.

2) This principle provides a method to solve the problem of determining a sequence $\underline{u}(0)$, $\underline{u}(1)$, ... $\underline{u}(N-1)$ which minimizes a performance criterion given by eq. (2-6).

Let $S_N[\underline{x}(0)]$ be the optimum value of S for an N-1 stage process starting at $k = 0$ with the initial state $\underline{x}(0)$.

Thus $S_{N-1}[x(1)]$ will be the optimum value of S for an N-1 stage process starting at $k = 1$ with the initial state $\underline{x}(1)$.

The mathematical translation of the principle of optimality can be written as:

$$S_N[\underline{x}(0)] = \min_{\underline{u}(0)} \left[S_{N-1}(\underline{x}(0)) + F_o(0)\right] \tag{3-8}$$

The method will consist in a " stage by stage optimization".

Consider first a one stage process. The performance criterion is

$$S_1 \left[\underline{x}(0) \right] = \min_{\underline{u}(0)} \left[F_N(0) \right] \tag{3-9}$$

This equation determines $\underline{u}(0)$ for a one stage process; it will be $\underline{u}(1)$ for a two stage process, $\underline{u}(2)$ for a three stage process, $\underline{u}(N-1)$ for a N stage process.

Consider now a two stage process; $\underline{u}(1)$ has been determined by the preceeding step.

To determine $\underline{u}(0)$ consider the expression, derived from the principle of optimality

$$S_2 \left[\underline{x}(0) \right] = \min_{\underline{u}(0)} \left[F_{N-1}(0) + S_1 (x(1)) \right] \tag{3-10}$$

In this expression, it is possible to replace $\underline{x}(1)$ by its expression given by eq. (3-1)

$$\underline{x}(1) = f\left[\underline{x}(0), \underline{u}(0) \right] \tag{3-11}$$

Then $S_2 \left[\underline{x}(0) \right]$ depends only on $\underline{x}(0)$ which is known as an initial condition, and on $\underline{u}(0)$. It is then possible to determine the optimal $\underline{u}(0)$. The process can be repeated N times, and each step will determine a member of the optimal sequence $\underline{u}(0)$, $\underline{u}(1)$, ... $\underline{u}(1)$, ... $\underline{u}(N-1)$.

3-4 OPTIMIZATION IN STATE-SPACE-

Consider a system given by Eqs. (2-8) and (2-9). The problem is to determine a sequence of control inputs $\underline{u}(0)$, $\underline{u}(1)$, ... $\underline{u}(N-1)$ which minimizes a performance criterion of the form of equations (2-4) or (2-5).

In these expressions for S, it is always possible to substitute $\underline{v}(k)$ by the way of equation (2-9) and to get, for the performance criterion, an expression involving the state vector and the control input. It is then possible to carry on an optimization process, just by considering a plant with an output vector $\underline{x}$, a control input $\underline{u}$ and a

dynamic behaviour behaviour characterized by equation (2-8).

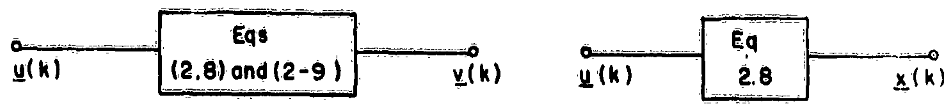


FIG. 3-2. STATE-VECTOR CONSIDERED AS THE NEW OUTPUT OF A SYSTEM

This is referred to as " optimization in state-space" . It is the type of optimization that will be considered in section IV and V.

dynamic behaviour behaviour characterized by equation (2-8).

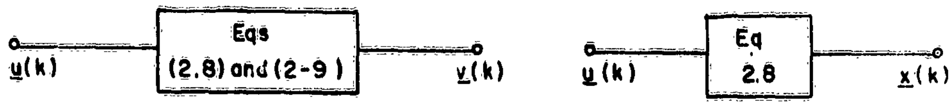


FIG. 3-2. STATE-VECTOR CONSIDERED AS THE NEW OUTPUT OF A SYSTEM

This is referred to as " optimization in state-space". It is the type of optimization that will be considered in section IV and V.

SECTION IV

OPTIMIZATION WITH DESIRED FINAL VALUE-

4-1 - INTRODUCTION-

Let a multivariable sampled-data syst^m be represented by

$$\underline{x}(k+1) = A\,\underline{x}(k) + B\,\underline{u}(k) \tag{4-1}$$

$$\underline{v}(k) = C\,\underline{x}(k) + D\,\underline{u}(k) \tag{4-2}$$

$\underline{u}(k)$: p dimensional input vector

$\underline{v}(k)$ q dimensional output vector

$\underline{x}(k)$ x dimensional state vector

A : n row, n column matrix which can be made symetrical (cf section II)

B : q row n column matrix

D : q row p column matrix.

Let $\underline{r}(k)$ be a given q dimensional reference input vector.

Consider now an N stage process where

1) At the end of the process the output $\underline{v}$ should be as close as possible to the reference input $\underline{r}$.

2) The average value of the magnitude of $\underline{u}$, during the process should be as small as possible. A convenient performance criterion will be

$$S = \left[v(N) - r(N)\right]^T \left[v(N) - r(N)\right] + \sum_{k=0}^{N} u^T(k)\,u(k) \tag{4-3}$$

The problem is to determine a sequence of inputs $\underline{u}(0)$, $\underline{u}(1)$, ... $\underline{u}(N-1)$ so that S is minimum.

The problem depends of course on a certain number of initial conditions. If the order of the system is n, the state-vector has n components and the system depends generally on n initial conditions (the case where this statement is not true will be examined later). It will be assumed that the initial state is given by the components of the state-vector at the instant zero. In other words $\underline{x}(0)$ is assumed to be a known vector.

In the case where the components of the state vector do not depend on $\underline{u}$ (this is the case when the driving function depends on $\underline{u}(k)$ only) the initial state can be given by the components of the vector outputs at instants 0, 1, 2, ... m, m depending on the respective values of n and p. If $n = m\,p + a$ with $a < p$ and m and a integers $\underline{x}(0)$ will depend an $\underline{v}(0)$, $\underline{v}(1)$, ... $\underline{v}(m)$ and a components of $\underline{v}(m + 1)$.

As a sample example consider a nth order single -input, single output system

$$v(k + n) + a_{n-1}\ v(k + n - 1) + \ldots + a_o\, v(k) = b_o\, u(n) \qquad (4\text{-}4)$$

A set of appropriate state variables are

$$x_1(k) = v(k)$$

$$x_2(k) = v(k+1)$$

$$\vdots$$

$$x_n(k) = v(k + n - 1)$$

$$\underline{x}(0) = \begin{bmatrix} v(0) \\ v(1) \\ v(2) \\ \vdots \\ v(n-1) \end{bmatrix}$$

It is important to notice, that, in the case of a sampled-data system, the "initial state" of the system does not mean, like in the continuous case, the state of the system at the instant zero, but can require, for instance, the knowledge of the output at a later instant. It is clear that equation (4-4) defines the variable $v(k)$ under the condition that $v(0)$, $v(1)$, ... $v(k-1)$ are given as "initial conditions".

S will be minimized by the method of dynamic programming discussed in Section III. Let $S_N\left[\underline{x}(o)\right]$ be the minimum value of S for an N stage process which starts at $k = o$ with $\underline{x}(o)$ specified. The different S's are related by the following relations

$$S_1\left[\underline{x}(o)\right] = \min_{\underline{u}(o)} \left\{\left[\underline{v}(o) - r(o)\right]^T\left[\underline{v}(o) - \underline{r}(o)\right] + \underline{u}^T(o)\underline{u}(o)\right\} \qquad (4\text{-}5)$$

$$S_2\left[\underline{x}(o)\right] = \min_{\underline{u}(o)} \left\{S_1\left[\underline{x}(1)\right] + \underline{u}^T(o)\underline{u}(o)\right\} \qquad (4\text{-}6)$$

$$S_3\ \underline{x}(o) = \min_{\underline{u}(o)} \left\{S_2\left[\underline{x}(1)\right] + \underline{u}^T(o)\underline{u}(o)\right\} \qquad (4\text{-}7)$$

$$\vdots$$

$$S_N\ \underline{x}(o) = \min_{\underline{u}(o)} \left\{S_{N-1}\left[\underline{x}(1)\right] + \underline{u}^T(o)\underline{u}(o)\right\} \qquad (4\text{-}8)$$

4.2. ONE STAGE PROCESS

The optimum value of $\underline{u}(o)$ in this case is given by eq. (4-5)

$$S_1\left[\underline{x}(o)\right] = \min_{\underline{u}(o)} \left\{ \left[\underline{v}(o) - \underline{r}(o)\right]^T\left[\underline{v}(o) - \underline{r}(o)\right] + \underline{u}^T(o)\underline{u}(o)\right.$$

or else

$$S_1\left[\underline{x}(o)\right] = \min_{\underline{u}(o)} \left\{\left[C\underline{x}(o) + D\underline{u}(o) - \underline{r}(o)\right]^T\left[C\underline{x}(o) + D\underline{u}(o) - \underline{r}(o)\right] + \underline{u}^T(o)\underline{u}(o)\right\} \qquad (4\text{-}9)$$

Eq. (4-9) is an expression involving all the components of $\underline{u}(o)$. Letting all the derivatives $\frac{\partial S_1}{\partial u_1(o)}$, $\frac{\partial S_2}{\partial u_2(o)} \cdots \frac{\partial S_1}{\partial u_p(o)}$ equal to zero and condensing the result in matrix form, $\underline{u}_1^o(o)$ (i.e., $\underline{u}(o)$ optimum for a one stage process) is given by (cf appendix II)

$$D^T \left[C \underline{x}(o) + D \underline{u}_1^o(o) - \underline{r}(o) \right] + \underline{u}_1^o(o) = o \tag{4-10}$$

or

$$\underline{u}_1^o = U_1 \left[\underline{r}(o) - C\underline{x}(o) \right] \tag{4-11}$$

where

$$U_1 = (I + D^T D)^{-1} D^T \tag{4-12}$$

In this case the result can be made particularly simple by noticing that, according to Eq. (4-2)

$$\underline{y}(o) = C\underline{x}(o) + D\underline{u}(o)$$

Equation (4-10) can be written as

$$D^T \left[\underline{y}(o) - \underline{r}(o) \right] + \underline{u}_1^o(o) = o \tag{4-13}$$

Hence

$$\underline{u}_1^o(o) = D^T \left[\underline{r}(o) - \underline{y}(o) \right] \tag{4-14}$$

Consequently, the controller, in the case of a one stage process can be synthesized by simple feedback. This will be true for the last stage of any multi-stage process.

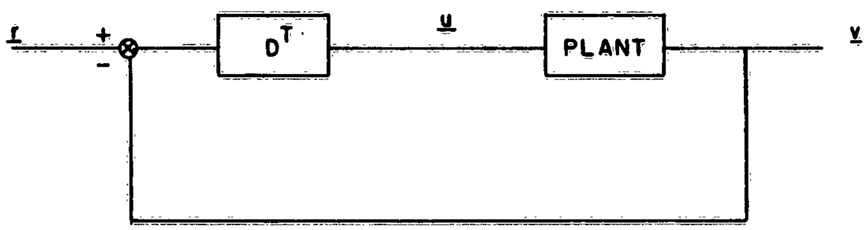


FIGURE 4-1. CONTROLLER FOR A ONE STAGE PROCESS

The actual minimum value of $S_1\left[\underline{x}(o)\right]$ is then:

$$S_1^o\left[\underline{x}(o)\right] = \left[C\underline{x}(o) + D U_1 (\underline{r}(o) - C\underline{x}(o)) - \underline{r}(o) \right]^T \left[C\underline{x}(o) + D U_1 (\underline{r}(o) - C\underline{x}(o)) - \underline{r}(o) \right]$$
$$+ \left[\underline{r}(o) - C\underline{x}(o) \right] U_1^T U_1 \left[\underline{r}(o) - C\underline{x}(o) \right] \tag{4-15}$$

Let

$$H_1 = (D\,U_1 - I)^T (D\,U_1 - I) + U_1^T U_1 \qquad (4\text{-}16)$$

H_1 is a symetrical matrix which depends only on the matrix D

$S_1\left[\underline{x}(o)\right]$ can be written in the form

$$S_1^o\left[\underline{x}(o)\right] = \left[\underline{r}(o) - C\,\underline{x}(o)\right]^T H_1\left[\underline{r}(o) - C\,\underline{x}(o)\right] \qquad (4\text{-}17)$$

Which shows that $S_1\left[\underline{x}(o)\right]$ is a linear form of the vector $\underline{r}(o) - C\,\underline{x}(o)$

In the rather frequent case where $D = O$, U_1 is the zero matrix and H_1 is the identity matrix

$$U_1 = O \qquad\qquad H_1 = I$$

and $S_1^o\left[\underline{x}(o)\right]$ optimum is equal to the magnitude of the vector $\underline{r}(o) - C\,\underline{x}(o)$

4.3. TWO STAGE PROCESS

The error criterion for a two stage process is

$$\begin{aligned} S_2\left[\underline{x}(o)\right] &= S_1\left[\underline{x}(1)\right] + u^T(o)\; u(o) \\ &= \left[\underline{r}(1) - C\,\underline{x}(1)\right]^T H_1\left[\underline{r}(1) - C\,x(1)\right] + \underline{u}^T(o)\,\underline{u}(o) \end{aligned} \qquad (4\text{-}18)$$

Equation (4-1) for $k = o$ shows that

$$\underline{x}(1) = A\,\underline{x}(o) + D\,\underline{u}(o) \qquad (4\text{-}19)$$

Hence

$$\begin{aligned} S_2\left[\underline{x}(o)\right] = &\left[\underline{r}(1) - C\,A\,\underline{x}(o) - C\,B\,\underline{u}(o)\right] H_1\left[\underline{r}(1) - C\,A\,\underline{x}(o) - C\,B\,\underline{u}(o)\right] \\ &+ \underline{u}^T(o)\;\underline{u}(o) \end{aligned} \qquad (4\text{-}20)$$

The result shown in appendix II gives $\underline{u}_2^o(o)$ by the equation

$$-(CB)^T H_1\left[\underline{r}(1) - C\,A\,\underline{x}(o) - C\,B\;\underline{u}_2^o(o)\right] + \underline{u}_2^o(o) = o$$

or

$$\underline{u}_2^o(o) = U_2\left[\underline{r}(1) - C\,A\,\underline{x}(o)\right] \qquad (4\text{-}21)$$

$$U_2 = \left[I + C B^T H_1 C B \right] \; C B^T H_1 \tag{4-22}$$

The minimum value of the performance criterion for two stages is

$$S_2^o\left[\underline{x}(o)\right] = \left[\underline{r}(1) - CA\underline{x}(o) - C B U_2 (\underline{r}(1) - CA\underline{x}(o))\right]^T H_1 \left[\underline{r}(1) - CA\underline{x}(o) - C B U_2 (\underline{r}(1) - CA\underline{x}(o))\right]$$

$$+ \left[\underline{r}(1) - C A \underline{x}(o)\right]^T U_2^T U_2 \left[\underline{r}(1) - C A x(o)\right]$$

or

$$S_2\left[\underline{x}(o)\right] = \left[\underline{r}(1) - CA\underline{x}(o)\right]^T H_2 \left[\underline{r}(1) - CA\underline{x}(o)\right] \tag{4-23}$$

where

$$H_2 = \left[I - C B U_2 \right]^T H_1 \left[I - C B U_2 \right] + U_2^T U_2 \tag{4-24}$$

In equation (4-23), H_2 is a symetrical matrix.

The synthesis of the controller requires the knowldege a priori, of $\underline{r}(1)$. The value of $\underline{u}_2^o(o)$ can be then precalculated, but the controller cannot be synthesized by feedback.

In the case where the components of the state-vector are either uncontrollable or unobservable, then $C B \equiv O$, therefore $\underline{u}_2^o = \underline{o}$.

This result could have been easily predicted. In this case the v's cannot be influenced by the u's. So S will be minimum for $\underline{u}^T(o)\ \underline{u}(o) = o$ i. e., $\underline{u}(o) = o$.

CONCLUSION

The input which optimizes a one stage process is

$$\underline{u}_1^o = U_1\left[\underline{r}(o) - C\underline{x}(o)\right]$$

The sequence of input which optimizes a two stage process is

$$\underline{u}_2^o(o) = U_2\left[\underline{r}(1) - C A\underline{x}(o)\right]$$

$$\underline{u}_2^o(1) = U_1\left[\underline{r}(1) - C\ \underline{x}(1)\right]$$

The U's are p rows q column matrices depending on A, B, C, D.

$S_1^o\left[\underline{x}(o)\right]$ is a linear form of the vector $\underline{r}(o) - C\underline{x}(o)$. $S_2^o\left[\underline{x}(o)\right]$ is a linear form of the vector $\underline{r}(o) - C A \underline{x}(o)$.

Those results can be generalized for an N stages process.

4.4. N STAGE PROCESS

Assume that for an N-1 stage process the error criteron is

$$S_{N-1}\left[\underline{x}(o)\right] = \left[\underline{r}(N-2) - C A^{N-2}\underline{x}(o)\right] H_{N-1}\left[\underline{r}(N-2) - C A^{N-2}\underline{x}(o)\right] \tag{4-25}$$

where H_{N-1} is a q x q symetrical matrix.
For an N stage process the error criterion is

$$S_N\left[\underline{x}(o)\right] = S_{N-1}\left[\underline{x}(1)\right] + \underline{u}^T(o)\,\underline{u}(o)$$

$$= \left[\underline{r}(N-1) - C A^{N-2}\underline{x}(1)\right]^T H_{N-1}\left[\underline{r}(N-1) - CA^{N-2}\underline{x}(1)\right] + \underline{u}^T(o)\,\underline{u}(o)$$

or else

$$S_N\left[\underline{x}(o)\right] = \left[\underline{r}(N-1) - CA^{N-1}\underline{x}(o) - CA^{N-2}B\underline{u}(o)\right]^T H_{N-1}\left[\underline{r}(N-1) - CA^{N-1}\underline{x}(o) - CA^{N-2}B\underline{u}(o)\right]$$
$$+ \underline{u}^T(o)\,\underline{u}(o) \tag{4-26}$$

The result shown in appendix II gives the equation giving $\underline{u}^o_N(o)$

$$-(CA^{N-2}B)^T H_{N-1}\left[\underline{r}(N-1) - CA^{N-1}\underline{x}(o) - CA^{N-2}B\,\underline{u}^o_N(o)\right] + \underline{u}^o_N(o) = o$$

or

$$\underline{u}^o_N(o) = U_N\left[\underline{r}(N-1) - C A^{N-1}\underline{x}(o)\right] \tag{4-27}$$

where

$$U_N = \left[I + (C A^{N-2}B)^T H_{N-1}(C A^{N-2}B)\right]^{-1}(C A^{N-2}B)^T H_{N-1} \tag{4-28}$$

This shows that $\underline{u}^o_N(o)$ is a linear function of $\underline{r}(N-1)$ and of $\underline{x}(o)$. Therefore $\underline{r}(N-1)$ has to be known before the process starts.

The minimum value of the performance criterion is then

$$S^o_N\left[\underline{x}(o)\right] = \left[\underline{r}(N-1) - CA^{N-1}\underline{x}(o)\right] H_N\left[r(N-1) - CA^{N-1}\underline{x}(o)\right] \tag{4-29}$$

where

$$H_N = \left[I - C A^{N-2}B\,U_N\right] H_{N-1}\left[I - CA^{N-2}B\,U_N\right] + U_N^T U_N \tag{4-30}$$

Equation (4-30) can be interpreted as a recursion formula between the H's with the initial value

$$H_1 = \left[I + D^T D\right]^{-1}$$

Therefore it is possible to compute all the H's using eq. (4-30) and deduce the U's via eq. (4-28).

This work can be programmed in advance on a computer and the U's can be known *a priori*.

CONCLUSION

For an N stage process, the sequence of optimal inputs is

$$\underline{\underline{u}}_N^o(o) = U_N \left[\underline{\underline{r}} (N-1) - CA^{N-1} \underline{\underline{x}} (o) \right] \tag{4-31}$$

$$\underline{\underline{u}}_N^o(1) = U_{N-1} \left[r (N-1) - CA^{N-2} \underline{\underline{x}} (1) \right] \tag{4-32}$$

$$\underline{\underline{u}}_N^o(2) = U_{N-2} \left[r (N-1) - CA^{N-3} \underline{\underline{x}} (2) \right] \tag{4-33}$$

$$\vdots$$

$$\underline{\underline{u}}_N^o(N-1) = U_1 \left[\underline{\underline{r}} (N-1) - C \underline{\underline{x}} (N-1) \right]$$

$\underline{\underline{x}}$ (o) being known, $\underline{\underline{u}}_N^o$ can be computed; and this cannot generally be done by feedback for $\underline{\underline{x}}$ (o) can depend on the state of the system at later instants. A synthesis, using feedback will then be possible in the only case where the order of the system is equal to the rank of the output.

For the next step, $\underline{\underline{x}}$ (1) is given by

$$\underline{\underline{x}} (1) = A \underline{\underline{x}} (o) + B \underline{\underline{u}}_1^o (o)$$

Eq. (4-32) gives then $\underline{\underline{u}}_N^o$ (1), and so on.

The sequence of optimal inputs has to be precalculated, and fed into the plant at the proper instants.

All the expressions involve A to a certain power. Thus it is advisable to treat the problem in the state variables which diagonalize A.

If

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & O \\ & & \ddots & \\ O & & & \lambda_N \end{bmatrix} \quad (n \times n)$$

$$A^N = \begin{bmatrix} \lambda_1^N & & & \\ & \lambda_2^N & & \\ & & \ddots & \\ & & & \lambda_N^N \end{bmatrix} \quad (n \times n) \tag{4-34}$$

4.5. CASE WHERE SOME COMPONENTS OF THE STATE-VECTOR ARE UNCONTROLLABLE OR UNOBSERVABLE.

The coefficient of $\underline{x}$ (o) in Eq. (4-27) is CA^{N-1}. If some of the coordinates are unobservable C has $n^c + n^f$ columns of zeros (which can always be the $n^c + n^f$ last ones.

C can be written as:

$$C = \begin{bmatrix} \underbrace{\begin{bmatrix} C_{ij} \end{bmatrix}}_{n'} & \underbrace{\begin{matrix} O & O \\ O & O \\ O & O \\ O & O \end{matrix}}_{n^c + n^f} \end{bmatrix} \tag{4-35}$$

where $n' = n^* + n^o$ = number of observable coordinates. On the other hand, A^{N-1} is a diagonal matrix given by

$$A^{N-1} = \begin{bmatrix} \lambda_1^{N-1} & & & \\ & \lambda_2^{N-1} & & O \\ & & \ddots & \\ O & & & \lambda_n^{N-1} \end{bmatrix} \tag{4-36}$$

Consequently

$$CA^{N-1} = \left.\left[\begin{array}{cc|cc} C_{11}\lambda_1^{N-1} & C_{n'n'}\lambda_{n'}^{N-1} & 0 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \\ C_{1n}\lambda_1^{N-1} & C_{n'n}\lambda_{n'}^{N-1} & 0 & 0 \end{array}\right]\right\} n \qquad \begin{array}{c} \underbrace{\hspace{5cm}}_{n'} \quad \underbrace{\hspace{1.5cm}}_{n^c+n^f} \end{array} \tag{4-37}$$

The $n^c + n^f$ last columns of CA^{N-1} are zeros. As a consequence <u>$\underline{u}$ (o) does not depend on the unobservable coordinates of $\underline{x}$ (o).</u>

This result could have been predicted. The unobservable coordinates, by definition, do not affect the output; so they do not appear in the expression of S. As a result, the optimization can be carried on systems S^* and S^o and system S^c and S^f can be forgotten. The order of the system is $n^* + n^o$ and the problem depends on $n^* + n^o$ initial conditions, which are the components of vector $\underline{y}^*(o)$ and $\underline{y}^o(o)$.

Consider now the equation $\underline{y} = \underline{y}^* + \underline{y}^o$ which states that the actual output of the system is equal to the sum of the "controllable and observable" part and of the observable but uncontrollable part. Saying that $\underline{y}$ should be as close as possible to a reference input $\underline{r}$ is equivalent to saying that $\underline{y}^*$ should be as close as possible to a reference

$$\underline{r}' = \underline{r} - \underline{y}^o \tag{4-38}$$

$\underline{y}^o$ depends only on the vector $y^o(o)$ <u>but is independant of $\underline{u}$.</u> It is thus possible to determine $\underline{y}^o$ and therefore $\underline{r}'$ a priori, before starting any optimization process.

<u>CONCLUSION</u>

Instead of carrying on an optimization process over an $n^* + n^o$ system with a reference input $\underline{r}$, it is possible to carry out the optimization on the n^*th order system S^*, with a reference input $\underline{r}' = \underline{r} - \underline{y}^o$

FIGURE 4-2. <u>OPTIMIZATION ON SYSTEM S*</u>

Finally the case where all coordinates are either uncontrollable or unobservable (no system S*) has already been mentioned. In this case the output is disconnected from the input and the control problem has no solution. The only thing to do to minimize S is to set $\underline{u} = o$.

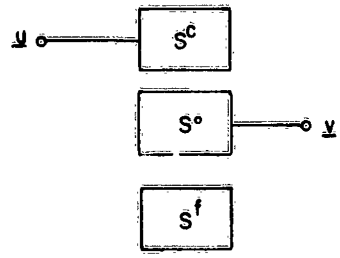


FIGURE 4-3. CASE WHERE ALL COORDINATES ARE EITHER UNCONTROLLABLE OR UNOBSERVABLE.

4.6. EXAMPLES

Example 1

Consider a two inputs two outputs system as shown in fig. 4-4 characterized by the equations

$$\left\{\begin{array}{l} v_1(k+2) + 3\,v_1(k+1) - 2\,v_2(k) = u_1(k) \\ v_1(k+1) + 3\,v_1(k) + v_2(k+1) + 3\,v_2(k) = u_2(k) \end{array}\right\} \qquad \begin{array}{r}(4\text{-}39)\\(4\text{-}40)\end{array}$$

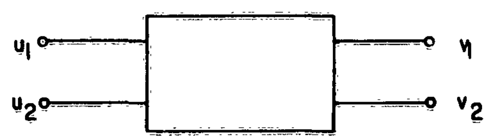


FIGURE 4-4. TWO INPUTS, TWO OUTPUTS SYSTEM

A set of appropriate state-variables is

$$\begin{aligned} \underline{x}_1(k) &= v_1(k) \\ \underline{x}_2(k) &= v_1(k+1) \\ \underline{x}_3(k) &= v_2(k) \end{aligned} \qquad (4\text{-}41)$$

$$\underline{v}(k) = \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix} \qquad \underline{u}(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

The initial conditions are stated by the vector

$$\underline{x}(o) = \begin{bmatrix} v_1(o) \\ v_1(1) \\ v_2(o) \end{bmatrix} \qquad (4\text{-}42)$$

The dynamics of the plant are defined by the equations

$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 2 \\ -3 & -1 & -3 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}(k) \qquad (4\text{-}43)$$

$$\underline{v}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(k)$$

The normal coordinates are defined by

$$\underline{x}(k) = \rho\, \underline{y}(k)$$

where

$$\rho = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -3 \\ -1 & 1 & 0 \end{bmatrix} \qquad (4\text{-}44)$$

The plant dynamics are defined by the new set of equations:

$$\underline{y}(k+1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \underline{y}(k) + \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \underline{u}(k) \qquad (4\text{-}45)$$

$$\underline{y}(k) = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} y(k) \qquad (4\text{-}46)$$

The system is supposed initially relaxed $\underline{x}(o) = \underline{y}(o) = \underline{O}$

1. STAGE PROCESS

As $D = O$ $\quad U_1 = O$ $\quad H_1 = I$

$$\underline{u}_1^o(o) = O$$

2. STAGE PROCESS

U_2 is given by eq. (4-22).

$$C\,B = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & 1 \end{bmatrix}$$

$$(C\,B)^T = \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}$$

$$(C\,B)^T H_1 (C\,B) = \begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix} \qquad (4\text{-}47)$$

$$U_2 = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{5}{14} \end{bmatrix}$$

$$\underline{u}_2(o) = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{5}{14} \end{bmatrix} \underline{r}(1)$$

for $\quad \underline{y}(o) = \underline{O}$

$\underline{u}_2^o(1) = O$

If the process has more than two stages it is necessary to compute H_2, given by equation (4-24). The result is

$$H_2 = \begin{bmatrix} \frac{130}{49} & -\frac{57}{98} \\ -\frac{57}{98} & \frac{41}{98} \end{bmatrix} \tag{4-48}$$

Example 2

Consider the single-input, single-output, second order system

$$v(k+2) - \frac{1}{4} v(k) = u(k+1) \tag{4-49}$$

The state vector can be chosen of the form

$$x_1(k) = v(k) \tag{4-50}$$

$$x_2(k) = v(k+1) - F_1 u(k) \tag{4-51}$$

then

$$x_1(k+1) = x_2(k) + F_1 u(k) \tag{4-52}$$

$$x_2(k+1) = \frac{1}{4} x_1(k) + F_2 u(k) \tag{4-53}$$

F_1 and F_2 are determined by writing

$$x_2(k+1) = v(k+2) - F_1 u(k+1) = \frac{1}{4} v(k) + F_2 u(k)$$

$$v(k+2) = \frac{1}{4} v(k) + F_1 u(k+1) + F_2 u(k) \tag{4-54}$$

On the other hand

$$v(k+2) = \frac{1}{4} v(k) + u(k+1) \tag{4-55}$$

Comparing eq. (4-54) and (4-55) gives F_1 and F_2

$$F_1 = 1 \qquad F_2 = 0$$

Finally the state vector is

$$\underline{x}(k) = \begin{bmatrix} v(k) \\ \\ v(k+1) - u(k) \end{bmatrix} \tag{4-56}$$

and the plant dynamics can be written as

$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{4-57}$$

$$\underline{v}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(k) \tag{4-58}$$

The normal coordinates can be defined as

$$\underline{x} = \begin{bmatrix} 1 & 1 \\ \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \underline{y} \tag{4-59}$$

and the normal form representation is

$$\underline{y}(k+1) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \underline{y}(k) + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} u(k)$$

$$\underline{v}(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{y}(k)$$

<u>One stage process.</u>

As the transmission matrix D = O the optimal input is (cf 4-2)

$$\underline{u}_1^o = 0$$

<u>Two stage process.</u>

$\underline{u}_2^o$ is given by eq. (4-21) and (4-22)

$$CB = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \frac{3}{2}$$

eq. (4-22) gives

$$U_2 = \frac{3/2}{1 + (\frac{3}{2})^2} = \frac{6}{13}$$

and equation (4-22) gives

$$u_2^o\ (o) = \frac{6}{13}\left[r\ (1) - \frac{1}{2}\ y_1\ (o) + \frac{1}{2}\ y_2\ (o)\right] \tag{4-60}$$

$$u_2^o\ (1) = 0 \tag{4-61}$$

Three stages process.

Eq. (4-30) gives $H_2 = \frac{81}{169}$

and Eq. (4-28) gives

$$U_3 = \frac{CAB\ H_2}{1 + (CAB)^2\ H_2} \qquad CAB = -\frac{1}{4}$$

$$U_3 = 0.12$$

$$u_3^o\ (o) = 0.12\left[r\ (2) - \frac{1}{4}\ y_1\ (o) - \frac{1}{4}\ y_2\ (o)\right] \tag{4-62}$$

$$u_3^o\ (1) = \frac{6}{13}\left[r\ (2) - \frac{1}{2}\ y_1\ (o) + \frac{1}{2}\ y_2\ (o)\right] \tag{4-63}$$

$$u_3^o\ (2) = 0 \tag{4-64}$$

Example 3

Case where some components are uncontrollable or unobservable.

Let a third order one-dimensional input, one-dimensional output system be given by

$$\underline{x}\ (k+1) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}\ (k) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \underline{u}\ (k) \tag{4-65}$$

$$v\ (k) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \underline{x}\ (k) \tag{4-66}$$

The system is initally relaxed.

The third coordinate is uncontrollable, the second unobservable.

The system S^o is characterized by

$$x_3\ (k+1) = x_3\ (k)$$

$$v^o\ (k) = x_3\ (k)$$

Therefore $v^{o}(k+1) = v^{o}(k) = v^{o}(o) = o$

$$r' = r - v^{o} = r$$

The optimization process can be carried out using the equations

$$x_1(k+1) = \frac{1}{2}\, x_1(k)$$
$$v(k) = x_1(k)$$

Which are equivalent to

$$v(k+1) = \frac{1}{2}\, v(k) \tag{4-67}$$

In this case an optimization in the state-space is equivalent to an optimization with respect to the actual output.

Example 4

Consider the cascade connection of the two systems

$$S_a \begin{cases} y_a(k+1) = -\, y_a(k) + u_a(k) \\ v_a(k) = y_a(k) + u_a(k) \end{cases} \tag{4-68}$$

$$S_b \begin{cases} y_b(k+1) = -2\, y_b(k) + u_b(k) \\ v_b(k) = y_b(k) - u_b(k) \end{cases} \tag{4-69}$$

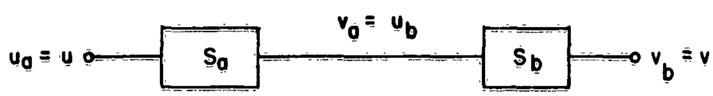


FIG. 4-5. CASCADE CONNECTION OF TWO SYSTEMS

The state vector of the overall system S can be taken as

$$x_1 = y_a \qquad x_2 = y_b$$

The system S can be represented by

$$\underline{x}(k+1) = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \tag{4-70}$$

$$v(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} \underline{x}(k) - u(k) \tag{4-71}$$

and for

$$\rho = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The normal form representation is

$$\underline{y}(k+1) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \underline{y}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$v(k) = \begin{bmatrix} 0 & -1 \end{bmatrix} \underline{y}(k) - u(k)$$

All coordinates are either uncontrollable or unobservable. Even though S_a and S_b spearately can be optimized, their cascade connection is such that input and output are disconnected and the optimization problem is meaningless.

SECTION V

REGULATOR PROBLEM

5.1. Introduction

In this section the regulator problem is considered that is, the output $\underline{v}$ is desired to follow as closely as possible a reference input $\underline{r}$ and besides $\underline{u}$ should be as "small" as possible.

The expression to be minimized is

$$S = \sum_{k=o}^{n} \left[\underline{v}(k) - \underline{r}(k) \right]^T \left[\underline{v}(k) - \underline{r}(k) \right] + \underline{u}^T(k)\, \underline{u}(k) \tag{5-1}$$

The plant is still described by the set of equations discussed in Section II

$$\begin{aligned} \underline{x}(k+1) &= A\, \underline{x}(k) + B\, \underline{u}(k) \\ v(k) &= C\, x(k) + D\, u(k) \end{aligned} \tag{5-2}$$

The initial conditions are given by the componants of the state-vector at instant o

$$\underline{x}(o) = \begin{bmatrix} x_1(o) \\ x_2(o) \\ \\ x_{n-1}(o) \end{bmatrix} \tag{5-3}$$

Let $S_N \left[\underline{x}(o) \right]$ be defined, as explained in Section III, as the minimum value of S for an N stages process starting at k = o with the initial value $\underline{x}(o)$. The set of following equations can be written

$$\underline{S}_1 \left[\underline{x}(o) \right] = \min_{\underline{u}(o)} \left\{ \left[\underline{v}(o) - \underline{r}(o) \right]^T \left[\underline{v}(o) - \underline{r}(o) \right] + \underline{u}^T(o)\, \underline{u}(o) \right\} \tag{5-4}$$

$$S_2 \left[\underline{x}(o) \right] = \min_{\underline{u}(o)} \left\{ S_1 \left[\underline{x}(1) \right] + \left[\underline{v}(o) - \underline{r}(o) \right]^T \left[\underline{v}(o) - \underline{r}(o) \right] + \underline{u}^T(o)\, \underline{u}(o) \right\} \tag{5-5}$$

$$S_N\left[\underline{x}(o)\right] = \min_{\underline{u}(o)} \left\{ S_{N-1}\left[\underline{x}(1)\right] + \left[\underline{v}(o) - \underline{r}(o)\right]^T \left[\underline{v}(o) - \underline{r}(o)\right] + \underline{u}^T(o)\,\underline{u}(o) \right\} \tag{5-6}$$

The optimization process will be, as explained in section III, the so-called "dyanamic programming" process.

5.2. One stage process

The expression to be minimized in the case of a one-stage process is

$$S_1\left[\underline{x}(o)\right] = \min_{\underline{u}(o)} \left\{ \left[\underline{v}(o) - \underline{r}(o)\right]^T \left[\underline{v}(o) - \underline{r}(o)\right] + \underline{u}^T(o)\,\underline{u}(o) \right\} \tag{5-7}$$

The minimization of this expression has already been studied in 3-2. In this case the controller can be synthesized by simple feedback and the optimal input is given by

$$\underline{u}_1^o = U_1\left[\underline{r}(o) - c\,\underline{n}(o)\right] \tag{5-8}$$

Where
$$U_1 = \left[I + D^T D\right] D^T$$

or

$$\underline{u}_1^o = D^T\left[\underline{r}(o) - \underline{v}(o)\right]$$

The minimum value of $S_1\left[\underline{x}(o)\right]$ is given by

$$S_1\left[\underline{x}(o)\right] = \left[\underline{r}(o) - C\,\underline{x}(o)\right]^T H_1 \left[\underline{r}(o) - C\,\underline{x}(o)\right] \tag{5-9}$$

Where H_1 is a symetrical matrix.

2.3. Two stage process

The expression to minimize is given by Eq. (5-5)

$$S_2\left[\underline{x}(o)\right] = \left[\underline{r}(1) - C\underline{x}(1)\right]^T H_1\left[\underline{r}(1) - C\underline{x}(1)\right] + \left[\underline{v}(o) - \underline{r}(o)\right]^T\left[\underline{v}(o) - \underline{r}(o)\right]^T + \underline{u}^T(o)\underline{u}(o) \tag{5-10}$$

or using eq (5-2)

$$S_2\left[\underline{x}(o)\right] = \left[\underline{r}(1) - CA\,\underline{x}\,(o) - CB\,\underline{u}(o)\right]^T H_1\left[\underline{r}(1) - CA\,\underline{x}(o) - CB\,\underline{u}\,(o)\right]$$
$$+\left[C\,\underline{x}(o) + D\underline{u}\,(o) - \underline{r}(o)\right]^T\left[C\,\underline{x}(o) + D\,\underline{u}(o) - \underline{r}(o)\right] + \underline{u}^T(o)\,\underline{u}\,(o) \qquad (5\text{-}11)$$

The computation given in Appendix III shows that $\underline{u}_2^{\,o}(o)$ is given by the equation

$$(-CB)^T H_1\left[\underline{r}\,(1) - CA\,\underline{x}\,(o) - CB\,\underline{u}\,(o)\right] + D^T\left[C\,\underline{x}\,(o) + D\,\underline{u}\,(o) - \underline{r}(o)\right] + \underline{u}\,(o) = o \qquad (5\text{-}12)$$

or

$$\underline{u}_2^{\,o}\,(o) = U_x^{(2)}\,\underline{x}\,(o) + U_{r_o}^{(2)}\,\underline{r}\,(o) + U_{r_1}^{(2)}\,\underline{r}\,(1) \qquad (5\text{-}13)$$

where

$$U_x^{(2)} = -\left[(CB)^T H_1\,CB + H_1\right]^{-1}\left[(CB)^T H_1\,CA + D^T C\right] \qquad (5\text{-}14)$$

$$U_{ro}^{(2)} = \left[(CB)^T H_1\,CB + H_1\right]^{-1} D^T \qquad (5\text{-}15)$$

$$U_{r_1}^{(2)} = -\left[(CB)^T H_1\,CB + H_1\right]^{-1}(CB)^T H_1\,CA \qquad (5\text{-}16)$$

The result can be written under a more condensed form by introducing the 2q dimensional vector

$$\underline{R}_2\,(o) = \begin{bmatrix}\underline{r}\,(o)\\ \underline{r}\,(1)\end{bmatrix} \qquad (5\text{-}17)$$

The index 2 means that the vector is 2q dimensional.

More generally the vectors $R_K\,(o)$, $R_K\,(1)$. . are defined as

$$\underline{\underline{R}}_K(o) = \begin{bmatrix} \underline{r}(o) \\ \underline{r}(1) \\ \\ \underline{r}(K) \end{bmatrix} \quad \text{Kq dimensional}$$

$$\underline{\underline{R}}_K(1) \quad \begin{bmatrix} \underline{r}(1) \\ \underline{r}(2) \\ \underline{r}(K+1) \end{bmatrix} \tag{5-18}$$

Let

$$U_R^{(2)} = \begin{bmatrix} U_{r_o}^{(2)} & U_{r_1}^{(2)} \end{bmatrix} \tag{5-19}$$

Eq. (5-13) can be rewritten as

$$\underline{\underline{u}}_2^o(o) = U_x^{(2)}\, \underline{\underline{x}}(o) + U_R^{(2)}\, \underline{\underline{R}}_2(o) \tag{5-20}$$

Similarly Eq. (5-8) shows that

$$\underline{\underline{u}}_2^o(1) = U_x^{(1)}\, \underline{\underline{x}}(1) + U_R^{(1)}\, \underline{\underline{R}}_1(o) \tag{5-21}$$

where

$$U_x^{(1)} = -U_1\, C$$

$$U_R^{(1)} = U_1$$

It is now possible to compute the minimum value of $S_2\ \underline{\underline{x}}(o)$

$$\begin{aligned} S_2\left[\underline{x}(o)\right] = &\left[\underline{r}(1) - CA\,\underline{x}(o) - CB\,(U_x^{(2)}\underline{x}(o) + U_R^{(2)}\ \underline{\underline{R}}_2(o))\right]^T H_1 \left[\qquad\qquad\right] \\ &+ \left[C\,\underline{x}(o) + D\,(U_x^{(2)}\,\underline{x}(o) + U_R^{(2)}\ \underline{\underline{R}}_2(o) - \underline{r}(o)\right]^T \left[\qquad\qquad\right] \\ &+ \left[U_x^{(2)}\ \underline{x}(o) + U_R^{(2)}\ \underline{\underline{R}}_2(o)\right]^T \left[\qquad\right] \end{aligned} \tag{5-22}$$

This expression is a linear form of the two vectors $\underline{x}(o)$ and $\underline{\underline{R}}_2(o)$. It can be written as follows

$$S_2\left[\underline{x}(o)\right] = \underline{x}^T(o)\, H_2\, \underline{x}(o) + \underline{R}_2^T(o)\, K_2\, \underline{R}_2(o) + \underline{x}^T(o)\, L_2\, \underline{R}_2(o) \qquad (5\text{-}23)$$

Where H_2, K_2, L_2 are symmetrical matrices depending on A, B, C, D.

3-4. Three stage process

The expression to be minimized in this case is

$$S_3\left[\underline{x}(o)\right] = \underline{x}^T(1)\, H_2\, \underline{x}(1) + \underline{R}_2^T(1)\, K_2\, \underline{R}_2(1) + \underline{x}^T(1)\, L_2\, R_2(1)$$
$$+\left[\underline{v}(o) - \underline{r}(o)\right]^T\left[\underline{v}(o) - \underline{r}(o)\right] + \underline{u}^T(o)\, \underline{u}(o) \qquad (5\text{-}24)$$

or if $\underline{x}(1)$ is substituted by the way of Eq. (5-2)

$$S_3\left[\underline{x}(o)\right] = \left[A\,\underline{x}(o) + B\,\underline{u}(o)\right]^T H_2\left[A\,\underline{x}(o) + B\,\underline{u}(o)\right] + \underline{R}_2^T(1)\, K_2\, R_2(1) \qquad (5\text{-}25)$$
$$+\left[A\,\underline{x}(o) + B\,\underline{u}(o)\right]^T L_2\underline{R}_2(1) + \left[C\,\underline{x}(o) + D\underline{u}(o) - \underline{r}(o)\right]^T\left[C\underline{x}(o) + D\underline{u}(o) - \underline{r}(o)\right] + \underline{u}^T(o)\underline{u}(o)$$

Therefore $\underline{u}_3^o(o)$ is given by the following equation (cf Appendix III)

$$2\, B^T H_2\left[A\,\underline{x}(o) + B\underline{u}(o)\right] + B^T K_2\underline{R}_2(1) + D^T\left[C\,\underline{x}(o) + D\,\underline{u}(o) - \underline{r}(o)\right] + \underline{u}(o) = o \qquad (5\text{-}26)$$

or

$$\underline{u}_3^o(o) = -\left[I + 2B^T H_2 B + D^T D\right]^{-1}\left[(2B^T H_2 A + D^T C)\,\underline{x}(o) + B^T K_2\underline{R}_2(1) + D^T\underline{r}(o)\right] \qquad (5\text{-}27)$$

This can be rearranged as

$$\underline{u}_3^o(o) = U_x^{(3)}\,\underline{x}(o) + U_R^{(3)}\,\underline{R}_3(o) \qquad (5\text{-}28)$$

Substituting this value of $\underline{u}_3^o(o)$ in the expression of $S_3\left[\underline{x}(o)\right]$ it is clear that $S_3^o\left[\underline{x}(o)\right]$ is only a function of $\underline{x}(o)$ and $\underline{R}_3(o)$ for

$$\underline{\underline{R}}(1) = 2q \left\{ \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \right. \underline{\underline{R}}_3(o) \tag{5-29}$$

$$\underbrace{}_{q}\;\underbrace{}_{q}$$

So $S_3\left[\underline{\underline{x}}(o)\right]$ can be written in the form

$$S_3\left[\underline{\underline{x}}(o)\right] = \underline{\underline{x}}^T(o)\, H_3\, \underline{\underline{x}}(o) + \underline{\underline{R}}_3{}^T(o) K_3\, \underline{\underline{R}}_3(o) + \underline{\underline{x}}^T(o)\, L_3\, \underline{\underline{R}}_3(o) \tag{5-30}$$

And the process can continue in the same way.

5-5. CONCLUSIONS

1) For an N stages process the sequence of optimal input can be written in the form

$$\begin{aligned} \underline{\underline{u}}_N^o(o) &= U_x^{(N)}\, \underline{\underline{x}}(o) + U_R^{(N)}\, \underline{\underline{R}}_N(o) \\ \underline{\underline{u}}_N^o(1) &= U_x^{(N-1)}\, \underline{\underline{x}}(1) + U_R^{(N-1)}\, \underline{\underline{R}}_{N-1}(1) \\ \underline{\underline{u}}_N(N-1) &= U_x^{(1)}\, \underline{\underline{x}}(N-1) + U_R^{(1)}\, \underline{\underline{R}}_1(N-1) \end{aligned} \tag{5-31}$$

2) The general expression for the various matrices U's are too complicated to be given explicitely. However eqs (5-31) show that the value of the optimal input $\underline{u}^o$ (k) at the sampling instant k depends on $\underline{x}$ (k) and on the value of the reference $\underline{r}$ at all the sampling instants k, k + 1, . . . N following k.

3) The reference $\underline{r}$ must be known a priori. This makes the problem possible to solve when it is a regulator problem; but the dynamic problem where the system is desired to follow an indetermined reference input cannot be solved by this method.

4) The sequence of optimal inputs can be precalculated and fed into the plant. However the expressions involved are very complicated and the optimization became very impractical for high order systems and processes with a great number of stages.

5) Clearly the conclusions of Section IV about the effects of controllability and observability are still valid here, for those conclusions are valid for any type of criterion.

$$\underline{\underline{R}}(1) = 2q\left\{\left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]\right. \underline{\underline{R}}_3(o) \qquad (5\text{-}29)$$

(with column groups of width q and q)

So $S_3\left[\underline{x}(o)\right]$ can be written in the form

$$S_3\left[\underline{x}(o)\right] = \underline{x}^T(o)\, H_3\, \underline{x}(o) + \underline{\underline{R}}_3^T(o) K_3\, \underline{\underline{R}}_3(o) + \underline{x}^T(o)\, L_3\, R_3(o) \qquad (5\text{-}30)$$

And the process can continue in the same way.

5-5. CONCLUSIONS

1) For an N stages process the sequence of optimal input can be written in the form

$$\begin{aligned} \underline{u}_N^o(o) &= U_x^{(N)}\, \underline{x}(o) + U_R^{(N)}\, \underline{\underline{R}}_N(o) \\ \underline{u}_N^o(1) &= U_x^{(N-1)}\, \underline{x}(1) + U_R^{(N-1)}\, \underline{\underline{R}}_{N-1}(1) \\ \underline{u}_N(N-1) &= U_x^{(1)}\, \underline{x}(N-1) + U_R^{(1)}\, \underline{\underline{R}}_1(N-1) \end{aligned} \qquad (5\text{-}31)$$

2) The general expression for the various matrices U's are too complicated to be given explicitely. However eqs (5-31) show that the value of the optimal input $\underline{u}^o$ (k)at the sampling instant k depends on $\underline{x}$ (k) and on the value of the reference $\underline{r}$ at all the sampling instants k, k + 1, . . . N following k.

3) The reference $\underline{r}$ must be known a priori. This makes the problem possible to solve when it is a regulator problem; but the dynamic problem where the system is desired to follow an indetermined reference input cannot be solved by this method.

4) The sequence of optimal inputs can be precalculated and fed into the plant. However the expressions involved are very complicated and the optimization became very impractical for high order systems and processes with a great number of stages.

5) Clearly the conclusions of Section IV about the effects of controllability and observability are still valid here, for those conclusions are valid for any type of criterion.

5.6. **EXAMPLES**

Example 1

Let a second order single input single output system be given by

$$\underline{x}(N+1) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \underline{x}(N) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(N) \tag{5-32}$$

$$\underline{v}(N) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{n}(N)$$

The system is supposed initially relaxed.
The eigenvalues of matrix A are o and - 1 with

$$\rho = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{5-33}$$

The normal form representation is

$$\underline{y}(N+1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{y}(N) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(N) \tag{5-34}$$

$$v(N) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{y}(N)$$

It is clear that both coordinates are observable but only one is controllable.

The system S^o is defined by

$$y_2(N+1) = y_2(N)$$

$$v^o(N) = y_2(N) \tag{5-35}$$

$$v^o(N+1) = v^o(N) = v^o(o) \tag{5-36}$$

If the system is initially relaxed the new reference $r' = r - v^o$ is equal to $\underline{r}$.

The problem is now one-dimensional. The plant is defined by

$$v(N+1) = u(N) \tag{5-37}$$

and the criterion

$$S = \sum_{k=o}^{N} \left[v(k) - r(k)\right]^2 + u^2(k) \tag{5-38}$$

In this case the state vector is identical to the output v

One stage process

The results of section (5-2) show that

$$S_1\left[v(o)\right] = \left[v(o) - r(o)\right]^2 + u^2(o) \tag{5-39}$$

$$u_1^o(o) = o \tag{5-40}$$

$$S_1\left[v(o)\right] = \left[v(o) - r(o)\right]^2 \tag{5-41}$$

Two stage process

S_1 for a two-stage process stands as

$$\begin{aligned} S_2\left[v(o)\right] &= \left[v(1) - r(1)\right]^2 + \left[v(o) - r(o)\right]^2 + u^2(o) \\ &= \left[u(o) - r(1)\right]^2 + \left[v(o) - r(o)\right]^2 + u^2(o) \end{aligned} \tag{5-42}$$

$u_2^o(o)$ is given by the equation

$$u_2^o(o) - r(1) + u_2^o(o) = o \qquad \boxed{u_2^o(o) = \frac{r(1)}{2}} \tag{5-43}$$

and the minimum value of $S_3\left[v(o)\right]$ is

$$S_2\left[v(o)\right] = \frac{r^2(1)}{2} + \left[v(o) - r(o)\right]^2 \tag{5-44}$$

Three stages process

The expression to minimize is:

$$S_3 \left[v\,(o) \right] = \frac{r^2\,(2)}{2} + \left[v\,(1) - r\,(\text{[illegible]}) \right]^2 + \left[v\,(o) - r(\;) \right]^2 + u^2(o)$$

$$= \frac{r^2\,(2)}{2} + \left[u\,(o) - r\,(1) \right]^2 + \left[v(o) - r(o) \right]^2 + u^2\,(o) \qquad (5\text{-}45)$$

$u_3^o\,(o)$ is given by the equation

$$u_3^o\,(o) - r\,(1) + u_3^o\,(o) = o$$

$$\boxed{u_3^o\,(o) = \frac{r\,(1)}{2}} \qquad (5\text{-}46)$$

More generally, due to the fact that v (N + 1) does not depend on v (N) but only on u (N), for a N stages process, the sequence of optimal inputs is given by

$$\begin{aligned} u_N^o\,(o) &= \frac{r\,(1)}{2} \\ u_N^o\,(1) &= \frac{r\,(2)}{2} \\ u_N^o\,(2) &= \frac{r\,(3)}{2} \\ u_N^o\,(N-2) &= \frac{r\,(N-1)}{2} \\ u_N\,(N-1) &= o \end{aligned} \qquad (5\text{-}47)$$

Example 2

Let a first order single input single output system be given by

$$v(N+1) - v(n) = u(N+1) \tag{5-48}$$

If the state vector is chosen as

$$x(N) = v(N) - u(N) \tag{5-49}$$

the plant dynamics are given by

$$\begin{cases} x(N+1) = x(N) + u(N) \\ v(N) = x(N) + u(N) \end{cases} \tag{5-50}$$

One stage process

The expression to minimize is

$$S_1\left[x(o)\right] = \left[x(o) + u(o) - r(o)\right]^2 + u^2(o) \tag{5-51}$$

Therefore (cf Eq. 5-7)

$$\boxed{u_1^o(o) = \frac{1}{2}\ r(o) - x(o)} \tag{5-52}$$

and

$$S_1\left[x(o)\right] = \frac{1}{2}\left[r(o) - x(o)\right]^2 \tag{5-53}$$

Two stage process

The expression to minimize is

$$\begin{aligned} S_2\left[x(o)\right] &= \frac{1}{2}\left[r(1) - x(1)\right]^2 + \left[x(o) + u(o) - r(o)\right]^2 + u^2(o) \\ &= \frac{1}{2}\left[r(1) - x(o) - u(o)\right]^2 + \left[x(o) + u(o) - r(o)\right]^2 + u^2(o) \end{aligned} \tag{5-54}$$

and Eq. (5-13) gives the optimal inputs

$$\boxed{\begin{array}{l} u_2^o(o) = \frac{1}{5}\left[2\,r(o) + r(1) + 3\,x(o)\right] \\ \hline u_2^o(1) = \frac{1}{2}\left[r(1) - x(1)\right] \end{array}} \qquad (5\text{-}55)$$

SECTION VI

CONCLUSION

This study was concerned with multivariable sampled-data systems. It has been shown that an optimization of such a system with a squared error criterion is generally possible and leads to a sequence of optimal outputs depending 1) on the state-vector at any instant 2) on the reference input at sampling instants. The controller cannot work on a real time basis; the sequence of optimal inputs can be precalculated and fed directly into the plant. In the case of a desired final value studied in Section III, this precalculation is reasonably simple. The regulator type problem is much more involved and requires the knowledge of the reference input before the process starts.

The importance of controllability and observability of the coordinates of the state vector has been shown. In particular the optimization problem becomes meaningless when all the coordinates of the system are either controllable or unobservable. In the general case the number of initial conditions which has to be stated is equal to the number of observable coordinates. The optimization can always be carried out on a n^* th order system where n^* is the number of observable and controllable coordinates of the system.

In Section III the case of an optimization with desired final value has been studied quite thoroughly. The sequence of optimal outputs can be derived using a recursion formula. The regulator-type problem, studied in Section IV leads to rather involved results, and only the form of the results has been given. In both cases the study was limited to a finite number of stages.

APPENDIX I

Partitioning of a multivariable systems into subsystems according to controllability and observability.

Consider a multivariable n th order sampled-data* system S given by the equations.

$$\underline{y}(k+1) = \Lambda \underline{y}(k) + \beta \underline{u}(k) \qquad \text{(I-1)}$$

$$\underline{v}(k) = \gamma \underline{y}(k) + D \underline{u}(k) \qquad \text{(I-2)}$$

The order in which the components of the state-vector can be arranged is arbitrary. Let us arrange them so that

- The n^* first ($n^* \leq n$) are both controllable and observable -

Define $\underline{y}^*(k) = \begin{bmatrix} y_1(k) \\ y_{n^*}(k) \end{bmatrix}$

- The n^c following ($n^c \leq n$) are controllable and unobservable.

Define $\underline{y}^c(k) = \begin{bmatrix} y_{n^*+1}(k) \\ y_{n^*+n^c}(k) \end{bmatrix}$

- The n^o following ($n^o \leq n$) are observable and uncontrollable.

Define $\underline{y}^o(k) = \begin{bmatrix} y_{n^*+n^c+1}(k) \\ y_{n^*+n_c+n_o}(k) \end{bmatrix}$

- The n^f following ($n^f \leq n$) are neither controllable nor observable.

Define $y^f(k) = \begin{bmatrix} y_{n^+ + n_c^c + n^o + 1}(k) \\ y_n(k) \end{bmatrix}$

*We show here the theorem for a sampled-data system. It goes without saying that a similar theorem can be established for continuous systems.

Let now the different matrices be partitioned in the following way

$$\underline{y}(k) = \begin{bmatrix} \underline{y}^*(k) \\ \underline{y}^c(k) \\ \underline{y}^o(k) \\ \underline{y}^f(k) \end{bmatrix} \quad (I\text{-}3) \qquad \Lambda = \Big[\underbrace{\Lambda^*}_{n^*} \;\Big|\; \underbrace{\Lambda^c}_{n^c} \;\Big|\; \underbrace{\Lambda^o}_{n^o} \;\Big|\; \underbrace{\Lambda^f}_{n^f} \Big] \quad (I\text{-}4)$$

$$\beta = \begin{array}{c} \left[\begin{array}{c} \beta^* \\ \hline \beta^c \\ \hline \beta^o \\ \hline \beta^f \end{array}\right] \end{array} \begin{array}{l} \} \, n^* \\ \} \, n^c \\ \} \, n^o \\ \} \, n^f \end{array} \quad (I\text{-}5) \qquad \gamma = \Big[\underbrace{\gamma^*}_{n^*} \;\Big|\; \underbrace{\gamma^c}_{n^c} \;\Big|\; \underbrace{\gamma^o}_{n^o} \;\Big|\; \underbrace{\gamma^f}_{n^f} \Big] \quad (I\text{-}6)$$

By assumption the matrices β^c, β^f, γ^o, γ^f have all their elements equal to zero.

The system S can now be partitioned into the four following subsystems:

1) A subsystem S* defined by the equations

$$\underline{y}^*(k+1) = \Lambda^* \underline{y}^*(k) + \beta^* \underline{u}(k) \qquad (I\text{-}7)$$

$$\underline{v}^*(k) = \gamma^* \underline{y}^*(k) + D\,\underline{u}(k) \qquad (I\text{-}8)$$

β^* has no row of zeros and γ^* has no column of zeros. Therefore the system S* is both controllable and observable.

2) A subsystem S^c defined by the equations

$$\underline{y}^c(k+1) = \Lambda^c \underline{y}^c(k) + \beta^c \underline{u}(k) \qquad (I\text{-}9)$$

$$\underline{v}^c(k+1) = \gamma^c \underline{y}^c(k) \qquad (I\text{-}10)$$

γ^c has no column of zeros but all the rows of β^c are zeros. Therefore S^c is controllable but unobservable.

3) A subsystem S^o defined by the equations

$$\underline{y}^o (k+1) = \Lambda^o \underline{y}^o (k) + \beta^o \underline{u} (k) \tag{I-11}$$

$$\underline{v}^o (k) = \gamma^o \underline{y}^o (k) \tag{I-12}$$

This system is controllable but unobservable. $\gamma^o \equiv [o]$, $\underline{v}^o (k) = [o]$

4) A subsystem S^f defined by the equations

$$\underline{y}^f (k+1) = \Lambda^f \underline{y}^f (k) + \beta^f \underline{u} (k) \tag{I-13}$$

$$\underline{v}^f (k) = \gamma^f \underline{y}^f (k) \tag{I-14}$$

Obviously S^f is both unobservable and uncontrollable.

The preceeding analysis shows that $n = n^* + n^o + n^c + n^f$ (I-15)

$$\underline{v} (k) = \underline{v}^* (k) + \underline{v}^o (k) \tag{I-16}$$

for

$$\underline{v}^c (k) = \underline{v}^f (k) \equiv o$$

$$\underline{u}^* (k) = \underline{u}^c (k) = \underline{u} (k) \tag{I-17}$$

We notice too that we included the term D $\underline{u}$ (N) in the system S*. This is by no means necessary. It can be introduced with any of the subsystems, or even constitute a system by itself.

APPENDIX II

Consider the following expression

$$F = (A_1\underline{u}+\underline{x}_1)^T B_1 (A_1 y + \underline{x}_1) + (A_2\underline{u}+\underline{x}_2)^T B_2 (A_2\underline{u}+\underline{x}_2) + \ldots + (A_n\underline{u}+\underline{x}_n)^T B_n (A_n\underline{u}+\underline{x}_n) \tag{II-1}$$

Where $\underline{u}$ is a p dimensional vector

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix} \tag{II-2}$$

$\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n$ are $p_1, p_2, \ldots p_n$ dimensional vectors

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{1q_1} \end{bmatrix} \quad x_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{2q_2} \end{bmatrix} \ldots\ldots \quad x_n = \begin{bmatrix} x_{n1} \\ x_{n2} \\ \cdot \\ x_{nq_n} \end{bmatrix} \tag{II-3}$$

A_1 is a q_1 rows p_1 columns matrix

A_2 is a q_2 rows p_2 columns matrix

A_n is a q_n rows p_n columns matrix

B_1 is a q_1th order square symmetrical matrix
B_2 is a q_2th order square symmetrical matrix
B_n is a q_nth order square symmetrical matrix

Assuming that the x's are independent of $\underline{u}$ as all the matrixes involved in expression (II-1) the problem is to determine the vector $\underline{u}$ which minimizes F.

Solution

F is a linear combination of $x_1, x_2, \ldots x_n$. The minimum of F is obtained by solving the set of equations.

$$\frac{\partial F}{\partial u_1} = 0$$

$$\frac{\partial F}{\partial u_2} = 0 \qquad \text{(II-4)}$$

$$\vdots$$

$$\frac{\partial F}{\partial u_p} = 0$$

This is a set of p linear equations with p unknowns $u_1, u_2, \ldots u_p$.

Now let $F_1 = (A_1 \underline{u} + \underline{x}_1)^T B_1 (A_1 \underline{u} + \underline{x}_1)$

$$F_1 = \sum_{ijk\ell} b_{ij} (a_{ik} u_k + x_{1i}) (a_{j\ell} u_\ell + x_{1j}) \qquad \text{(II-5)}$$

$$i = 1, 2 \ldots q_1$$

$$j = 1, 2 \ldots q_1$$

$$k = 1, 2 \ldots p$$

$$\ell = 1, 2 \ldots p$$

$$\frac{\partial F_1}{\partial u_1} = \sum_{ij\ell} b_{ij} \; a_{i1} (a_{j\ell} u_\ell + x_{1j}) + \sum_{ijk} b_{ij} \, a_{j1} (a_{ik} u_k + x_{1i}) \qquad \text{(II-6)}$$

Notice that we can commute the indexes i and j or k and ℓ in any of the $\sum$

$$\frac{\partial F_1}{\partial u_1} = \sum_{ijk} (b_{ij} + b_{ji}) \, a_{i1} (a_{jk} u_k + x_{1j}) \qquad \text{(II-7)}$$

and as it is assumed $b_{ij} = b_{ij}$ (B_1 symmetrical)

$$\frac{\partial F_1}{\partial u_1} = 2 \sum_{ijk} a_{i1} \, b_{ij} \, (a_{jk} u_k + x_{1j}) = 0$$

similarly

$$\frac{\partial F_1}{\partial u_2} = 2 \sum a_{i2} \; b_{ij} \quad (a_{jk} \, u_k + x_{1j}) = o \qquad \text{(II-8)}$$

$$\frac{\partial F_1}{\partial u_p} = 2 \sum a_{ip} \, b_{ij} \, (a_{jk} \, u_k + x_{1j}) = o$$

This is equivalent to the matrix equation

$$A_1^T \left[B_1 \;\; A_1 \underline{\underline{u}} + \underline{\underline{x}}_1 \right] = o \qquad \text{(II-9)}$$

Coming back to the former expression of F it is clear that the solution of our problem is given by the equation

$$A_1^T \, B_1 \left[A_1 \underline{\underline{u}} + \underline{\underline{x}}_1 \right] + A_2^T \, B_2 \left[A_2 \underline{\underline{u}} + x_2 \right] + \ldots + A_n^T \, B_n \left[A_n \underline{\underline{u}} + x_n \right] = o \quad \text{(II-10)}$$

APPENDIX III

Appendix II gives the solution of the problem of minimizing with respect to $\underline{u}$ an expression

$$F = \sum_{k=1}^{n} (A_k \underline{u} + \underline{x}_k)^T B_k (A_k \underline{u} + \underline{x}_k) \tag{III-1}$$

This Appendix is concerned with a similar problem but the expression to minimize, instead of being as indicated in Eq. (III-1) has the form

$$F' = F + \sum_{k=1}^{m} (A_k \underline{u} + \underline{x}_k)^T B_k \underline{v}_k \tag{III-2}$$

Where $A_k \underline{u} + x_k$ is a p_k dimensional vector

$\underline{v}_k$ is a r_k dimensional vector

B_k is a p_k rows r_k columns matrix

Solution

$$\text{Let } F'' = \sum_{k=1}^{n} (A_k \underline{u} + \underline{x}_k)^T B_k \underline{v}_k \tag{III-3}$$

The minimum of F' is obtained by solving the set of equations

$$\frac{\partial F'}{\partial u_1} = \frac{\partial F}{\partial u_1} + \frac{\partial F''}{\partial u_1} = 0$$

$$\frac{\partial F'}{\partial u_2} = \frac{\partial F}{\partial u_2} + \frac{\partial F''}{\partial u_2} = 0 \tag{III-4}$$

$$\vdots$$

$$\frac{\partial F'}{\partial u_n} = \frac{\partial F}{\partial u_n} + \frac{\partial F''}{\partial u_n} = 0$$

$$\text{Let } F''_1 = (A_1 \underline{u} + \underline{x}_1)^T B_1 \underline{v}_1 \tag{III-5}$$

F''_1 can be written as

$$F''_1 = \sum_{ij} b_{ij} (a_{ik} u_k + x_{1i}) v_{ij} \tag{III-6}$$

$$i = 1, 2, \ldots p_1$$

$$j = 1, 2, \ldots r_1$$

The set of partial derivatives of F''_1 with respect of u_1, $u_2, \cdots u_p$ is given as

$$\frac{\partial F''_1}{\partial u_1} = \sum_{ij} b_{ij}\, a_{i1}\, v_{1j}$$

$$\frac{\partial F''_1}{\partial u_2} = \sum_{ij} b_{ij}\, a_{i2}\, v_{1j} \qquad \text{(III-7)}$$

$$\frac{\partial F''_n}{\partial u_p} = \sum_{ij} b_{ij}\, a_{in}\, v_{1j}$$

Letting all these partial derivatives to zero is equivalent to solving the matrix equation

$$\boxed{A_1^T\, B_1\, v_1 = o} \qquad \text{(III-8)}$$

Coming back to the expression of F' as given in (III-2) and taking into account the result given in (II-10), the $\underline{u}$ which solves our problem is given by the equation

$$2 \sum_{k=1}^{n} A_k^T\, B_k \left[A_k\, \underline{u} + \underline{x}_k\right] + \sum_{k=1}^{m} A^T_k\, B_k\, \underline{v}_k = o \qquad \text{(III-9)}$$

REFERENCES

1. Elmer G. Gilbert "Controllability and Observability in Multivariable Control Systems" presented at conference of Society for Industrial and Applied Mathematics, Nov. 1962 at Cambridge, Mass.

2. R.E. Kalman, Y.C. Ho, and K.S. Narendra "Controllability of Linear Dynamic Systems", Contribution to Differential Equations, Vol I, 1961.

3. Richard Bellman, "Introduction to Matrix Analysis", McGraw-Hill Book Co., Inc., New York 1960.

4. R.E. Kalman, and R.W. Koepcke, Poughkeepsie, N.Y. "Optimal Synthesis of Linear Sampling Control Systems Using Generalized Performance Indexes" ASME Paper No. 58 - IRD - 6.

5. R. Bellman, "Dynamic Programming", Princeton University, Princeton, New Jersey, 1957.

6. L.I. Rozonoer, "The Maximum Principle of Pantryagin in Optimal System Theory", Translated into English in Automation and Remote Control, June, July, August, 1960.